

# A Gram Determinant for Lickorish's Bilinear Form

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## Abstract

We use the Jones-Wenzl idempotents to construct a basis of the Temperley-Lieb algebra  $TL_n$ . This allows a short calculation for a Gram determinant of Lickorish's bilinear form on the Temperley-Lieb algebra.

**Keywords:** *Skein Theory, Temperley-Lieb Algebra.*

## 1 Introduction

In [W], Witten proposed the existence of 3-manifold invariants. A mathematically rigorous definition was given by Reshetikhin and Turaev [RT] using quantum groups and Kirby calculus [K]. Later, Lickorish [L1] provided an alternative proof by using a bilinear form on the Temperley-Lieb algebra  $TL_n$ . An important property Lickorish needed was that this bilinear form defined over  $\mathbb{Z}[A, A^{-1}]$  is degenerate at certain  $4(n+1)$ th roots of unity and nondegenerate at  $4i$ th roots of unity for  $i < n+1$ . Ko and Smolinsky obtained this result by using a recursive formula for the determinants of specific minors of this form [KS]. They did not give a closed form for the determinant. This was first done by Di Francesco, Golinelli and Guitter [FGG]. Di Francesco later gave a simpler proof. In this paper, we give a short derivation by using a skein-theoretic approach together with a combinatorial proposition from Di Francesco [F]. In order to do this, we construct a nice basis  $\mathfrak{D}_n$  for  $TL_n$ . In fact, there have been several bases of  $TL_n$  studied before. See [FGG], [F], or [GS]. It turns out that  $\mathfrak{D}_n$  is a rescaled version of the basis used in [F], but the properties of Jones-Wenzl idempotents significantly simplify the calculation. Our skein-theoretic approach is motivated by the colored graph basis for TQFT modules developed in Blanchet, Habegger, Masbaum, Vogel's paper [BHMV]. A skein theoretic derivation of a Gram determinant for the type B Temperley-Lieb algebra is given in [CP].

## 2 Temperley-Lieb Algebra

Let  $F$  be an oriented surface with a finite collection of points specified in its boundary  $\partial F$ . A link diagram in the surface  $F$  consists of finitely many arcs and closed curves in  $F$ , with a finite number of transverse crossings, each assigned over or under information. The endpoints of the arcs must be the specified points in  $\partial F$ . We define the skein of  $F$  as follows:

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**Definition 2.1.** Suppose  $A$  is a variable. Let  $\Lambda$  be the ring  $\mathbb{Z}[A, A^{-1}]$  localized by inverting the multiplicative set generated by elements of  $\{A^n - 1 \mid n \in \mathbb{Z}^+\}$ . The linear skein  $\mathcal{S}(F)$  is the module of formal linear sums over  $\Lambda$  of link diagrams in  $F$  quotiented by the submodule generated by the skein relations:

1.  $L \cup U = \delta L$ , where  $U$  is a trivial knot,  $L$  is a link in  $F$  and  $\delta = (-A^{-2} - A^2)$ ;
2.  $\times = A^{-1} \smile + A \succ$ .

Now, taking  $F$  to be the 2-disk  $D^2 = I \times I$ , we have:

**Definition 2.2.** The  $n^{\text{th}}$  Temperley-Lieb Algebra  $TL_n$  is the linear skein  $\mathcal{S}(D^2, n)$ , where  $n$  means there are  $n$  points specified in  $I \times \{0\}$  and  $I \times \{1\}$  respectively.

It is well known that  $TL_n$  has a basis, which consisting of non-crossing figures. We denote this basis by  $\mathfrak{B}_n$ . Some special elements  $\{1, e_1, \dots, e_{n-1}\}$  of the basis are shown in Figure 1. As an algebra,  $TL_n$  is generated by those special elements.

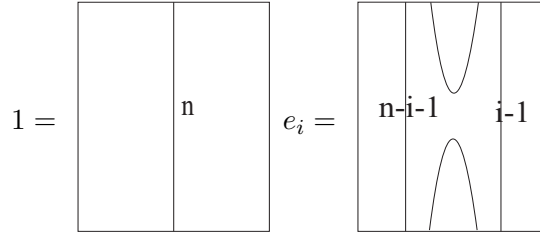


Figure 1: The integer  $i$  beside the arc means  $i$  parallel copies of the arc.

A significant property of this algebra in quantum invariant theory is that there is a natural bilinear form on  $TL_n$ . In [L1], Lickorish used this form to construct quantum invariants of 3-manifolds. We construct this bilinear form with respect to the basis  $\mathfrak{B}_n$  that we gave above:

**Definition 2.3.** Define a map on  $\mathfrak{B}_n \times \mathfrak{B}_n$  to  $\Lambda$  as follows:

$$G_n(s, r) = \langle \boxed{\mathbf{S}} \boxed{\mathbf{R}} \rangle ,$$

where  $s = \boxed{\mathbf{S}}$  and  $r = \boxed{\mathbf{R}}$  are elements in  $\mathfrak{B}_n$  and  $\langle, \rangle$  is the Kauffman bracket.

We extend this map to a bilinear form on  $TL_n$ , and still denote it by  $G_n$ . We denote the determinant of  $G_n$  with respect to  $\mathfrak{B}_n$  by  $\det(G_n)$ .

In this paper, we give a simple proof of the determinant of this bilinear form with respect to the basis  $\mathfrak{B}_n$ , which was also proved in [FGG]. The following is the main result.

**Theorem 2.4.**

$$\det(G_n) = \Delta_1^{c_n} \prod_{k=1}^n \left( \frac{\Delta_k}{\Delta_{k-1}} \right)^{\alpha_k}$$

where  $\Delta_i = \frac{(-1)^i (A^{2(i+1)} - A^{-2(i+1)})}{A^2 - A^{-2}}$ ,  $c_n = \frac{1}{n+1} \binom{2n}{n}$ , and  $\alpha_k = \binom{2n}{n-k} - \binom{2n}{n-k-1}$ .

**Remark 2.5.** From now on, we will use  $\text{card}$  to denote the cardinality of a set and  $\det$  the determinant of a matrix.

### 3 Properties of $TL_n$

In the 1990's, the properties of  $TL_n$  were studied by Lickorish [L2], Masbaum-Vogel [MV], Kauffman-Lins [KL] and some other people. Below we will summarize some results on  $TL_n$  that we will be using.

In this algebra, there is a sequence of idempotents, which are very important in constructing 3-manifold invariants. We will mainly use these idempotents to construct a basis for  $TL_n$ . They are defined as follows:

**Proposition 3.1.** *There is a unique element  $f_n \in TL_n$ , called  $n^{\text{th}}$  Jones-Wenzl idempotent, such that*

1.  $f_n e_i = 0 = e_i f_n$  for  $1 \leq i \leq n-1$ ;
2.  $(f_n - 1)$  belongs to the subalgebra generated by  $e_1, \dots, e_{n-1}$ ;
3.  $f_n f_n = f_n$ .

**Remark 3.2.** *We can put a box on the segment to denote the idempotent. But we will abbreviate the box from now on. Hence, we put an  $n$  beside the string to denote  $n$  parallel strings with an idempotent inserted, if otherwise is not stated. For example, we denote the figure on the left in Figure 2 by the figure on the right, which will be used frequently in this paper.*

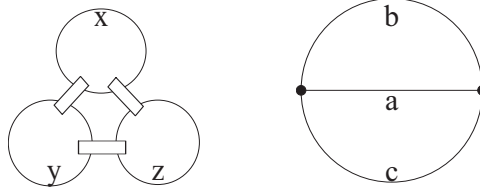


Figure 2: The left figure lies in  $S^2$ . The right figure is an abbreviation of the left one, where  $a = x + y, b = y + z, c = x + z$ . We denote the value of this diagram in  $\mathcal{S}(S^2)$  by  $\Theta(a, b, c)$ .

For the next property, we first set up some notation. Consider the skein space of the disc  $D$  with  $a + b + c$  specified points on its boundary. The points are partitioned into three sets of  $a, b, c$  consecutive points. The effect of adding the idempotents  $f_a, f_b, f_c$  just outside every diagram in such a disc with specified points is to map the skein space of the disc into a subspace of itself. We denote this subspace by  $T_{a,b,c}$ .

**Definition 3.3.** *The triple  $(a, b, c)$  of nonnegative integers will be called admissible if  $a + b + c$  is even,  $a \leq b + c$  and  $b \leq c + a$  and  $c \leq a + b$ .*

**Proposition 3.4.**

$$\dim(T_{a,b,c}) = \begin{cases} 0 & \text{if } a, b, c \text{ are not admissible;} \\ 1 & \text{if } a, b, c \text{ are admissible.} \end{cases}$$

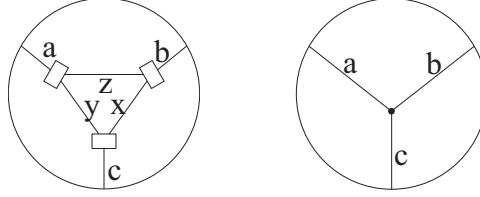


Figure 3: On the left is the generator of  $T_{a,b,c}$ . On the right is an abbreviation of the generator.

When  $(a, b, c)$  is admissible,  $T_{a,b,c}$  has a generator  $g$  on the left in Figure 3. We usually denote it in a simple way by the diagram on the right in the figure.

**Proposition 3.5.**

$$\text{Diagram with a loop labeled } a, b, c \text{ and external legs } a, d = \frac{\delta_{ad} \Theta(a, b, c)}{\Delta_a} \text{Diagram with a box labeled } a,$$

where  $\delta_{ad}$  is the Kronecker delta.

Similarly, consider the skein space of the disc  $D$  with  $a + b + c + d$  specified points on its boundary. The points are partitioned into four sets of  $a, b, c, d$  consecutive points. The effect of adding the idempotents  $f_a, f_b, f_c, f_d$  just outside every diagram in such a disc with specified points is to map the skein space of the disc into a subspace of itself. We denote this subspace by  $Q_{a,b,c,d}$ .

**Proposition 3.6.** A base for  $Q_{a,b,c,d}$  is the set of elements as in Figure 4, where  $j$  takes all values for which both  $(a, b, j)$  and  $(c, d, j)$  are admissible.

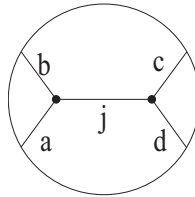


Figure 4: A basis element of  $Q_{a,b,c,d}$

**Proposition 3.7.**

$$\text{Diagram with two horizontal lines labeled } a, b = \sum \frac{\Delta_j}{\Theta(a, b, j)} \text{Diagram with two horizontal lines labeled } a, b \text{ and a central segment labeled } j$$

where the summation runs over all  $j$ 's such that  $(a, b, j)$  is admissible.

**Remark 3.8.** We denote  $\frac{\Theta(a,b,1)}{\Delta_a^2}$  by  $\Gamma(b,a)$ . It is easy to see that  $\Gamma(b,a) = 0$  if  $\|a - b\| > 1$ . Then Proposition 3.5 becomes

$$\begin{array}{c} \text{b} \\ \circlearrowleft \\ \text{a} \quad \text{d} \end{array} = \delta_{ad} \Gamma(b,a) \begin{array}{c} \text{a} \\ \boxed{\phantom{a}} \end{array} .$$

## 4 A Basis for the Temperley-Lieb Algebra from Jones-Wenzl Idempotents

Several bases of  $TL_n$  have been given before by, for example, [FGG] and [GS]. The idea of constructing the basis in this paper is motivated by [BHMV] and [L3]. They constructed bases for modules associated to surfaces by a certain topological quantum field theory. These bases were indexed by coloring of a trivalent graph in a handlebody.

**Definition 4.1.** Let  $D_{a_1, \dots, a_{2n-1}}$  be the element of  $TL_n$  in the Figure 5 where  $a_i$  satisfies:

1.  $a_1 = a_{2n-1} = 1$ ;
2.  $a_i \in \mathbb{N}$  for all  $i$ ;
3.  $\|a_i - a_{i-1}\| = 1$  for all  $i$ .

Let  $\mathfrak{A}_n$  be the collection of all  $n$ -tuples  $(a_1, \dots, a_{2n-1})$  satisfying the above conditions, and let  $\mathfrak{D}_n$  be the collection of all these  $D$ 's.

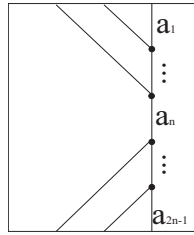


Figure 5: Each triple point is admissible.

**Lemma 4.2.** Suppose  $(a_1, \dots, a_{2n-1})$  and  $(b_1, \dots, b_{2n-1})$  satisfy all conditions above except  $a_1 = a_{2n-1} = 1$ . Then

$$\begin{aligned} & \langle D_{a_1, \dots, a_{2n-1}}, D_{b_1, \dots, b_{2n-1}} \rangle \\ &= \delta_{a_1 b_1} \dots \delta_{a_{2n-1} b_{2n-1}} \Gamma(a_1, a_2) \Gamma(a_2, a_3) \dots \Gamma(a_{2n-2}, a_{2n-1}) \Delta_{a_{2n-1}} . \end{aligned}$$

*Proof.* We prove the formula by induction.

When  $n = 2$ , by direct computation,

$$\begin{aligned} & \langle D_{a_1, a_2, a_3}, D_{b_1, b_2, b_3} \rangle \\ &= \delta_{a_1 b_1} \Gamma(a_1, a_2) \delta_{a_2 b_2} \Gamma(a_2, a_3) \delta_{a_3 b_3} \Delta_{a_3} . \end{aligned}$$

Thus the formula is true for  $n = 2$ . Now suppose the formula is true for  $n = k - 1$  and let  $n = k$ .

$$\begin{aligned}
& \langle D_{a_1, \dots, a_{2n-1}}, D_{b_1, \dots, b_{2n-1}} \rangle \\
&= \delta_{a_1 b_1} \Gamma(a_1, a_2) \delta_{a_{2n-1} b_{2n-1}} \Gamma(a_{2n-2}, a_{2n-1}) \langle D_{a_2, \dots, a_{2n-2}}, D_{b_2, \dots, b_{2n-2}} \rangle \\
&= \delta_{a_1 b_1} \Gamma(a_1, a_2) \delta_{a_{2n-1} b_{2n-1}} \Gamma(a_{2n-2}, a_{2n-1}) \times \\
&\quad \delta_{a_2 b_2} \dots \delta_{a_{2n-2} b_{2n-2}} \Gamma(a_2, a_3) \Gamma(a_3, a_4) \Gamma(a_{2n-3}, a_{2n-2}) \Delta_{a_{2n-2}} \\
&= \delta_{a_1 b_1} \dots \delta_{a_{2n-1} b_{2n-1}} \Gamma(a_1, a_2) \Gamma(a_2, a_3) \dots \Gamma(a_{2n-2}, a_{2n-1}) \Delta_{a_{2n-1}}.
\end{aligned}$$

Thus the formula holds for  $n = k$ . Hence, by induction, the formula holds.  $\square$

**Lemma 4.3.** *The elements of  $\{D_{a_1, \dots, a_{2n-1}}\}$  are orthogonal in  $TL_n$ , and so are linearly independent.*

*Proof.* This follows from Lemma 4.2.  $\square$

Now, we are going to prove that the elements of  $\mathfrak{D}_n$  generate  $TL_n$ . This can be proved using induction and Proposition 3.1 and 3.4. For variety, we give an alternative proof.

**Lemma 4.4.** *Each element of  $\{1, e_1, \dots, e_{n-1}\}$  can be expressed as a linear combination of  $D$ 's in  $\mathfrak{D}_n$ .*

*Proof.* We prove the lemma by induction and Proposition 3.7.

It is easy to see that the lemma is true for  $\mathfrak{D}_1, \mathfrak{D}_2$ .

Suppose the lemma is true for  $\mathfrak{D}_{n-1}$ , we need show that it is true for  $\mathfrak{D}_n$ .

For  $x \in \{1, e_2, e_3, \dots, e_{n-1}\}$ , we can obtain the result as in Figure 6. The proof for  $e_1$  is similar except at the second equality, we use Proposition 3.7 for each turn-back, see Figure 7.  $\square$

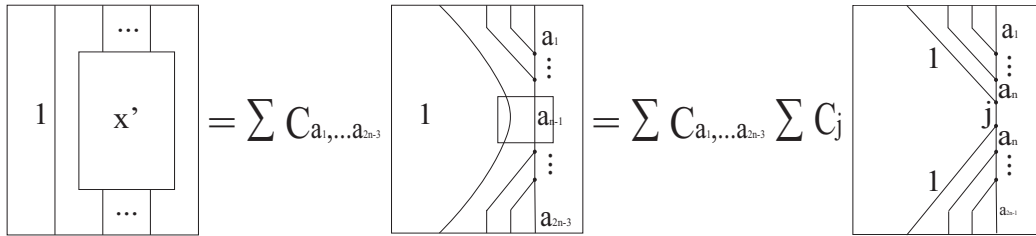


Figure 6:  $x'$ 's is a generator for  $TL_{n-1}$  by deleting the first arc in  $x$ . The first equality is from induction step. The second equality is from Proposition 3.7.

**Lemma 4.5.**  *$\mathfrak{D}_n$  is a basis of  $TL_n$ .*

*Proof.* Since each element in  $\mathfrak{B}_n$  can be written as a product of  $e_i$ 's, we can write each as a sum of products of elements in  $\mathfrak{D}_n$  by Lemma 4.4. Moreover, a product of elements in  $\mathfrak{D}_n$  can be written as a linear combination of elements in  $\mathfrak{D}_n$  by Proposition 3.5. So we can write elements in  $\mathfrak{B}_n$  as sums of elements in  $\mathfrak{D}_n$ . As  $\mathfrak{B}_n$  is a basis for  $TL_n$ , the lemma holds.  $\square$

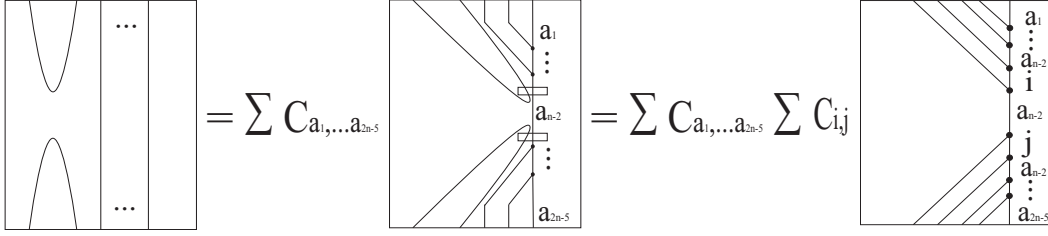


Figure 7: The proof is the same as in Figure 6 except we need use Proposition 3.7 twice.

## 5 Relation between $\mathfrak{B}_n$ and $\mathfrak{D}_n$

In this section, we will give a new system to denote the basis  $\mathfrak{B}_n$ . We draw a diagram similar to elements in  $\mathfrak{D}_n$  as in Figure 8, except we do not put idempotents on strings and we put an empty circle at each black triple point. If  $a_i = a_{i+1} + 1$ , we put Figure 9 in the corresponding circle. If  $a_i = a_{i+1} - 1$ , then we put Figure 10 in the corresponding circle. After filling all circles, we get a non-crossing diagram in  $TL_n$  for each sequence  $(a_1, \dots, a_{2n-1})$ , which satisfies the conditions in Definition 4.1. Those elements belong to  $\mathfrak{B}_n$ . Now, we give a total order on the set  $\mathfrak{A}_n$  as follows:  $(a_1, \dots, a_{2n-1}) < (b_1, \dots, b_{2n-1})$  if there is a  $j$  such that  $a_i = b_i$  for all  $i < j$  and  $a_j < b_j$ . This order on  $\mathfrak{A}_n$  induces an order on  $\{B_{a_1, \dots, a_{2n-1}}\}$  and  $\mathfrak{D}_n$  naturally. In this order, we will show that the representing matrix of  $\{B_{a_1, \dots, a_{2n-1}}\}$  with respect to basis  $\mathfrak{D}_n$  is upper triangular having 1's on the diagonal.

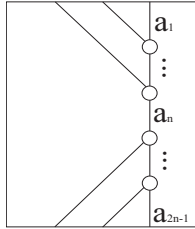


Figure 8: An example of  $B_{a_1, \dots, a_{2n-1}}$ .

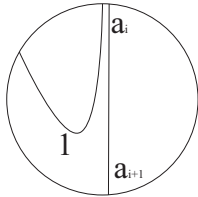


Figure 9:  $a_i = a_{i+1} + 1$

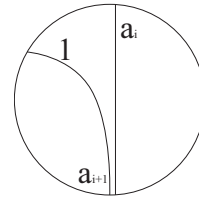


Figure 10:  $a_i = a_{i+1} - 1$

**Lemma 5.1.**  $\langle B_{a_1, a_2, \dots, a_{2n-1}}, D_{b_1, b_2, \dots, b_{2n-1}} \rangle = 0$  if  $(a_1, a_2, \dots, a_{2n-1}) < (b_1, b_2, \dots, b_{2n-1})$ .

*Proof.* Since  $(a_1, a_2, \dots, a_{2n-1}) < (b_1, b_2, \dots, b_{2n-1})$ , there is a  $j$  such that  $a_j < b_j$ . If we pair  $B_{a_1, a_2, \dots, a_{2n-1}}$  and  $D_{b_1, b_2, \dots, b_{2n-1}}$  together, we can find a circle passing through them at  $a_j$  and  $b_j$ . We cut the pairing along this circle to get an element as in Figure

11. By the properties of idempotents, it is easy to see that this element is 0 in  $\mathcal{S}(D^2)$  with  $a_j$  and  $b_j$  on the boundary. So we get the result.  $\square$

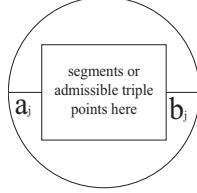


Figure 11: We have the idempotent at  $b_j$  and no idempotent at  $a_j$ .

Before we go on, we introduce a lemma and a corollary.

**Lemma 5.2.**  $\Theta(n, n+1, 1) = \Delta_{n+1}$ , and  $\Theta(n, n-1, 1) = \Delta_n$ .

*Proof.*

$$\begin{aligned}
& \Theta(n, n+1, 1) \\
&= \text{Diagram 1} = \text{Diagram 2} \\
&= \text{Diagram 3} = \text{Diagram 4} \\
&= \Delta_{n+1}
\end{aligned}$$

The diagrams are as follows:   
Diagram 1: A rectangle with a horizontal line at the top labeled '1', a horizontal line at the bottom labeled '1', a vertical line on the left labeled 'n', and a vertical line on the right labeled 'n+1'. Inside, there is a horizontal line labeled 'n+1'.   
Diagram 2: A rectangle with a horizontal line at the top labeled '1', a horizontal line at the bottom labeled '1', a vertical line on the left labeled 'n', and a vertical line on the right labeled 'n+1'. Inside, there are two horizontal lines labeled 'n+1' connected by vertical lines.   
Diagram 3: A rectangle with a horizontal line at the top labeled '1', a horizontal line at the bottom labeled '1', a vertical line on the left labeled 'n', and a vertical line on the right labeled 'n+1'. Inside, there are two horizontal lines labeled 'n+1' connected by vertical lines.   
Diagram 4: A rectangle with a horizontal line at the top labeled '1', a horizontal line at the bottom labeled '1', a vertical line on the left labeled 'n', and a vertical line on the right labeled 'n+1'. Inside, there are two horizontal lines labeled 'n+1' connected by vertical lines.

Similarly, it is easy to see that  $\Theta(n, n-1, 1) = \Delta_n$ .  $\square$

**Corollary 5.3.**

$$\Gamma(b, a) = \begin{cases} 1 & \text{if } a = b + 1, \\ \frac{\Delta_{a+1}}{\Delta_a} & \text{if } a = b - 1. \end{cases}$$

*Proof.* This follows easily from Lemma 5.2.  $\square$

Now, we can prove the following:

**Proposition 5.4.**  $\langle B_{a_1, a_2, \dots, a_{2n-1}}, D_{a_1, a_2, \dots, a_{2n-1}} \rangle = \langle D_{a_1, a_2, \dots, a_{2n-1}}, D_{a_1, a_2, \dots, a_{2n-1}} \rangle$  for all  $(a_1, a_2, \dots, a_{2n-1})$  in  $\mathfrak{A}_n$ .

*Proof.* We prove this by induction on  $n$ . Suppose  $n = 2$ . Then it is easy to check that

$$\langle B_{1,0,1}, D_{1,0,1} \rangle = \langle D_{1,0,1}, D_{1,0,1} \rangle, \langle B_{1,2,1}, D_{1,2,1} \rangle = \langle D_{1,2,1}, D_{1,2,1} \rangle.$$

Assume that the result is true for  $n < k$ . We will prove it is true for  $n = k$ . For  $(a_1, \dots, a_{2n-1}) = (1, 2, \dots, k, \dots, 2, 1)$ , we have

$$\langle B_{1,2,\dots,k,\dots,2,1}, D_{1,2,\dots,k,\dots,2,1} \rangle = \langle D_{1,2,\dots,k,\dots,2,1}, D_{1,2,\dots,k,\dots,2,1} \rangle$$



by direct computation. For  $(a_1, \dots, a_{2n-1}) \neq (1, 2, \dots, k, \dots, 2, 1)$ , we choose  $i > k > j$  such that  $a_{i-1} = a_{i+1}$ ,  $a_{j-1} = a_{j+1}$  and  $a_i = a_{i-1} + 1$ ,  $a_j = a_{j-1} + 1$ . Then by Proposition 3.5, we have

$$\begin{aligned} & \langle B_{a_1, a_2, \dots, a_{2n-1}}, D_{a_1, a_2, \dots, a_{2n-1}} \rangle \\ &= \Gamma(a_i, a_{i+1}) \Gamma(a_j, a_{j+1}) \\ & \quad \times \langle B_{a_1, \dots, \hat{a}_i, \hat{a}_{i+1}, \dots, \hat{a}_j, \hat{a}_{j+1}, \dots, a_{2n-1}}, D_{a_1, \dots, \hat{a}_i, \hat{a}_{i+1}, \dots, \hat{a}_j, \hat{a}_{j+1}, \dots, a_{2n-1}} \rangle \end{aligned}$$

where the hat on  $a_i$  means that it is removed.

It is easy to see that

$$B_{a_1, \dots, \hat{a}_i, \hat{a}_{i+1}, \dots, \hat{a}_j, \hat{a}_{j+1}, \dots, a_{2n-1}} \in \mathfrak{B}_{n-2}, D_{a_1, \dots, \hat{a}_i, \hat{a}_{i+1}, \dots, \hat{a}_j, \hat{a}_{j+1}, \dots, a_{2n-1}} \in \mathfrak{D}_{n-2}.$$

By Lemma 4.2 and Corollary 5.3,

$$\begin{aligned} & \langle D_{a_1, a_2, \dots, a_{2n-1}}, D_{a_1, a_2, \dots, a_{2n-1}} \rangle \\ &= \Gamma(a_i, a_{i+1}) \Gamma(a_j, a_{j+1}) \\ & \quad \times \langle D_{a_1, \dots, \hat{a}_i, \hat{a}_{i+1}, \dots, \hat{a}_j, \hat{a}_{j+1}, \dots, a_{2n-1}}, D_{a_1, \dots, \hat{a}_i, \hat{a}_{i+1}, \dots, \hat{a}_j, \hat{a}_{j+1}, \dots, a_{2n-1}} \rangle. \end{aligned}$$

By induction,

$$\begin{aligned} & \langle B_{a_1, \dots, \hat{a}_i, \hat{a}_{i+1}, \dots, \hat{a}_j, \hat{a}_{j+1}, \dots, a_{2n-1}}, D_{a_1, \dots, \hat{a}_i, \hat{a}_{i+1}, \dots, \hat{a}_j, \hat{a}_{j+1}, \dots, a_{2n-1}} \rangle \\ &= \langle D_{a_1, \dots, \hat{a}_i, \hat{a}_{i+1}, \dots, \hat{a}_j, \hat{a}_{j+1}, \dots, a_{2n-1}}, D_{a_1, \dots, \hat{a}_i, \hat{a}_{i+1}, \dots, \hat{a}_j, \hat{a}_{j+1}, \dots, a_{2n-1}} \rangle. \end{aligned}$$

Hence,

$$\langle B_{a_1, a_2, \dots, a_{2n-1}}, D_{a_1, a_2, \dots, a_{2n-1}} \rangle = \langle D_{a_1, a_2, \dots, a_{2n-1}}, D_{a_1, a_2, \dots, a_{2n-1}} \rangle.$$

□

**Proposition 5.5.**

$$\begin{pmatrix} B_{1,2,\dots,n,\dots,1} \\ \vdots \\ B_{1,0,1,0,\dots,0,1} \end{pmatrix} = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} D_{1,2,\dots,n,\dots,1} \\ \vdots \\ D_{1,0,1,0,\dots,0,1} \end{pmatrix}.$$

*Proof.* This follows easily from Lemma 5.1 and Proposition 5.4. □

**Corollary 5.6.**  $\{B_{a_1, \dots, a_{2n-1}}\} = \mathfrak{B}_n$ .

*Proof.* By Proposition 5.5, we can see that  $B_{a_1, \dots, a_{2n-1}} \neq B_{b_1, \dots, b_{2n-1}}$  if  $(a_1, \dots, a_{2n-1}) \neq (b_1, \dots, b_{2n-1})$ . Moreover,  $\text{card}\{B_{a_1, \dots, a_{2n-1}}\} = \text{card } \mathfrak{D}_n = \text{card } \mathfrak{B}_n$  and  $\{B_{a_1, \dots, a_{2n-1}}\} \subset \mathfrak{B}_n$ . So we have  $\{B_{a_1, \dots, a_{2n-1}}\} = \mathfrak{B}_n$ . □

By Corollary 5.6 and linear algebra, we can see that the  $\det(G_n)$  we get by using the basis  $\mathfrak{B}_n$  is the same as  $\det(G_n)$  we get by using the basis  $\mathfrak{D}_n$ .

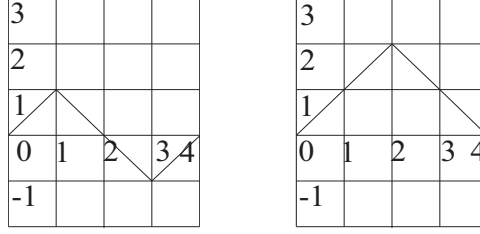


Figure 12: On the left is a lattice path. On the right is a Dyck path.

## 6 Lattice path

**Definition 6.1.** A lattice path in the plane is a path from  $(0,0)$  to  $(a,b)$  with northeast and southeast unit steps, where  $a, b \in \mathbb{Z}$ . A Dyck path is a lattice path that never goes below the  $x$ -axis. We denote the set of all Dyck path from  $(0,0)$  to  $(a,b)$  by  $\mathcal{D}_{(a,b)}$ .

**Remark 6.2.** There is a natural bijection  $f$  from  $\mathfrak{D}_n$  to  $\mathcal{D}_{(2n,0)}$ , the set of all Dyck paths from  $(0,0)$  to  $(2n,0)$  as follows:

For each  $D_{a_1, \dots, a_{2n-1}} \in \mathfrak{D}_n$ , we construct a path from  $(0,0)$  to  $(2n,0)$  with step  $(i, a_i)$  for all  $1 \leq i \leq 2n-1$ . Since  $a_i$  satisfies

1.  $a_1 = a_{2n-1} = 1$ ;
2.  $a_i \in \mathbb{N}$  for all  $i$ ;
3.  $\|a_i - a_{i-1}\| = 1$  for all  $i$ .

We can see that this is a Dyck path.

**Remark 6.3.** The reflection principle [C, page 22] says that the number of all Dyck paths from  $(0,0)$  to  $(2n,0)$  is the Catalan number  $C_n = \frac{1}{n+1} \binom{2n}{n}$ . Hence, we recover the well-known result that the dimension of  $TL_n$  is  $C_n$ .

## 7 Proof of the Main Theorem

Now we can start our proof of the main theorem. By Lemma 4.3, we know that  $\{D_{a_1, \dots, a_{2n-1}}\}$  is an orthogonal basis with respect to the bilinear form. Thus the matrix of  $G_n$  is a diagonal matrix under this basis. We have

$$\det(G_n) = \prod_{(a_1, \dots, a_{2n-1})} \langle D_{a_1, \dots, a_{2n-1}}, D_{a_1, \dots, a_{2n-1}} \rangle.$$

Then by Lemma 4.2, we have

$$\det(G_n) = \prod_{(a_1, \dots, a_{2n-1})} \Gamma(a_1, a_2) \Gamma(a_2, a_3) \dots \Gamma(a_{2n-2}, a_{2n-1}) \Delta_1.$$

Using Lemma 5.3, we can simplify  $\det(G_n)$  as follows:

Consider the tuple  $(D, i)$  such that  $D$  is an element of  $\mathfrak{D}_n$  and  $a_i = k$  in  $D$ . If  $a_{i+1} = k+1$ , then  $\Gamma(a_i, a_{i+1})$  is 1 by Lemma 5.3. So  $(D, i)$  will contribute 1 to  $\det(G_n)$ .

If  $a_{i+1} = k - 1$ , then  $\Gamma(a_i, a_{i+1}) = \frac{\Delta_{k+1}}{\Delta_k}$ . So  $(D, i)$  will contribute  $\frac{\Delta_k}{\Delta_{k-1}}$  to  $\det(G_n)$ . We denote by  $\mathfrak{S}_k$  the set of all tuple  $\{(D, i)\}$  with  $D \in \mathfrak{D}_n$  and  $a_i = k, a_{i+1} = k - 1$  in  $D$ . Let  $\alpha_k$  be the cardinality of  $\mathfrak{S}_k$ . Then

$$\det(G_n) = \Delta_1^{\text{card } \mathfrak{D}_n} \prod_{k=1}^n \left( \frac{\Delta_k}{\Delta_{k-1}} \right)^{\alpha_k}.$$

Now, the theorem is reduced to calculate  $\alpha_k$  for each  $k$ .

**Proposition 7.1.**

$$\alpha_k = \binom{2n}{n-k} - \binom{2n}{n-k-1}.$$

*Proof.* In Section 6, we already had a 1-1 correspondence  $f$  between  $\mathfrak{D}_n$  (the new basis we constructed) and  $\mathcal{D}_{(2n,0)}$  (Dyck paths from  $(0,0)$  to  $(2n,0)$ ). With respect to this correspondence, each pair  $(D, i)$  in  $\mathfrak{S}_k$  is associated to a pair  $(f(D), i)$  with  $a_i = k, a_{i+1} = k - 1$  in  $f(D)$ , that is the step from  $(i, a_i = k)$  to  $(i+1, a_{i+1} = k - 1)$  in the path  $f(D)$ . See Figure 13 for an example. Denote by  $\mathcal{S}_k$  the set of all pairs  $(f(D), i)$ , where  $f(D)$  has a step from  $k$  to  $k - 1$  at  $i$ . Thus we have a 1-1 correspondence between  $\mathfrak{S}_k$  and  $\mathcal{S}_k$ . Di Francesco [F, page 562] set up a 1-1 correspondence from  $\mathcal{S}_k$  to  $\mathcal{D}_{(2n,2k)}$ .

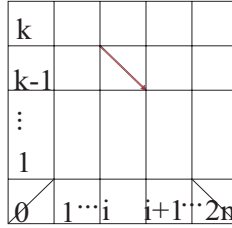


Figure 13: The bold step is a step going down from  $(i, a_i = k)$  to  $(i+1, a_{i+1} = k - 1)$ .

Then we have

$$\alpha_k = \text{card } \mathcal{D}_{(2n,2k)} = \binom{2n}{n-k} - \binom{2n}{n-k-1}.$$

For the convenience of reader, we now give Di Francesco's correspondence in our terminology. For an element  $\hat{P} \in \mathcal{D}_{(2n,2k)}$ , it should intersect the horizontal line  $y = k$  in a point  $p = (i, a_i = k)$  with  $a_{i+1} = k + 1$  and  $a_{i-1} = k - 1$  at least once. Let  $p$  be the rightmost such intersection. Now we cut  $\hat{P}$  at the point  $p$ , reflect the right part of  $\hat{P}$  with respect to  $y$ -axis, and shift it down by  $k$  units. Then we glue this part back to the left part. We get a Dyck path  $P$  from  $(0,0)$  to  $(2n,0)$ . In the resulting path  $P$ , we then choose the smallest  $i' \geq i$ , such that  $a_{i'} = k$  and  $a_{i'+1} = k - 1$ . We associate the path  $\hat{P}$  to the pair  $(P, i')$ . Therefore, we construct a map  $\phi : \mathcal{D}_{(2n,2k)} \rightarrow \mathcal{S}_k$ .

Conversely, for a pair  $(P, i)$ , where  $P \in \mathcal{D}_{(2n,0)}$ . We choose the largest  $i' \leq i$  with  $a_{i'} = k$  and  $a_{i'+1} = k - 1$ . We cut the path  $P$  at  $i'$ , reflect the right part with respect to the  $y$ -axis, shift up by  $n$  units and glue it back. Thus we construct a path  $\hat{P}$  in  $\mathcal{D}_{(2n,2k)}$ . We associate the tuple  $(P, i)$  to the path  $\hat{P}$ . Therefore, we construct a map  $\varphi : \mathcal{S}_k \rightarrow \mathcal{D}_{(2n,2k)}$ .

It is easy to see that  $\phi\varphi = id$  and  $\varphi\phi = id$ . Therefore, we have constructed a 1-1 correspondence between  $\mathcal{S}_k$  and  $\mathcal{D}_{(2n,2k)}$ .

By the reflection principle, we have

$$\text{card } \mathcal{D}_{(2n,2k)} = \binom{2n}{n-k} - \binom{2n}{n-k-1}.$$

Therefore, we have

$$\alpha_k = \binom{2n}{n-k} - \binom{2n}{n-k-1}.$$

□

## 8 Relation between $\mathfrak{D}_n$ and Di Francesco's second basis.

In this section, we extend our ring  $\Lambda$  to the complex numbers  $\mathbb{C}$  and let  $A$  be any non-zero complex number which is not a root of unity.

**Definition 8.1.**

$$ND_{a_1, \dots, a_{2n-1}} = \frac{D_{a_1, \dots, a_{2n-1}}}{\langle D_{a_1, \dots, a_{2n-1}}, D_{a_1, \dots, a_{2n-1}} \rangle^{\frac{1}{2}}}.$$

We call  $ND_{a_1, \dots, a_{2n-1}}$  the normalization of  $D_{a_1, \dots, a_{2n-1}}$ , and denote the normalized basis by  $\mathfrak{N}\mathfrak{D}_n$ .

**Theorem 8.2.**  $\mathfrak{N}\mathfrak{D}_n$  is the same basis as Di Francesco's second basis in  $[F]$ .

*Proof.* Di Francesco defined his orthonormal basis by a recursive equation [F, equation 3.19, Page 555]. So we just need to show that  $\mathfrak{N}\mathfrak{D}_n$  satisfies the recursive equation and the initial condition.

Let  $D_{a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{2n-1}}$  and  $D_{a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_{2n-1}}$  be two elements in  $\mathfrak{D}_n$  such that  $a_i = a_{i-1} - 1 = a_{i+1} - 1$  and  $a'_i = a_i + 2$ , that means they are equal everywhere except at  $i$ th arc. Then using the recursive formula for Jones-Wenzl idempotents at  $i$ th arc, we have

$$D_{a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{2n-1}} = D_{a_1, \dots, a_{i-1}, \hat{a}_i, a_{i+1}, \dots, a_{2n-1}} - \frac{\Delta_{a_i}}{\Delta_{a_i+1}} D_{a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{2n-1}}$$

where  $D_{a_1, \dots, a_{i-1}, \hat{a}_i, a_{i+1}, \dots, a_{2n-1}}$  is as in Figure 14. Since  $a_{i-1} = a_{i+1}$ , this is well defined.

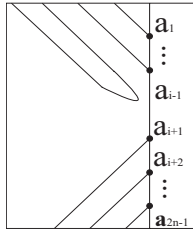


Figure 14:  $a_i$  disappears.

It is easy to see that  $D_{a_1, \dots, a_{i-1}, \hat{a}_i, a_{i+1}, \dots, a_{2n-1}} = e_i D_{a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{2n-1}}$ , where  $e_i$

acts on  $D_{a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{2n-1}}$  as in [F]. We divide the equation by the norm of  $D_{a_1, \dots, a_{i-1}, a_i', a_{i+1}, \dots, a_{2n-1}}$  on both sides. By Lemma 4.2, we have

$$\begin{aligned}
& ND_{a_1, \dots, a_{i-1}, a_i', a_{i+1}, \dots, a_{2n-1}} \\
&= \frac{(e_i D_{a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{2n-1}} - \frac{\Delta_{a_i}}{\Delta_{a_i+1}} D_{a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{2n-1}})}{(\Gamma(a_1, a_2) \dots \Gamma(a_{i-1}, a_i') \Gamma(a_i', a_{i+1}) \dots \Gamma(a_{2n-2}, a_{2n-1}) \Delta_{a_{2n-1}})^{\frac{1}{2}}} \\
&= (e_i - \frac{\Delta_{a_i}}{\Delta_{a_i+1}}) ND_{a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{2n-1}} \times \\
&\quad \frac{(\Gamma(a_1, a_2) \dots \Gamma(a_{i-1}, a_i) \Gamma(a_i, a_{i+1}) \dots \Gamma(a_{2n-2}, a_{2n-1}) \Delta_{a_{2n-1}})^{\frac{1}{2}}}{(\Gamma(a_1, a_2) \dots \Gamma(a_{i-1}, a_i') \Gamma(a_i', a_{i+1}) \dots \Gamma(a_{2n-2}, a_{2n-1}) \Delta_{a_{2n-1}})^{\frac{1}{2}}} \\
&= (e_i - \frac{\Delta_{a_i}}{\Delta_{a_i+1}}) ND_{a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{2n-1}} \frac{(\Gamma(a_{i-1}, a_i) \Gamma(a_i, a_{i+1}))^{\frac{1}{2}}}{(\Gamma(a_{i-1}, a_i') \Gamma(a_i', a_{i+1}))^{\frac{1}{2}}} \\
&= (e_i - \frac{\Delta_{a_i}}{\Delta_{a_i+1}}) ND_{a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{2n-1}} \frac{(\Gamma(a_{i-1}, a_i))^{\frac{1}{2}}}{(\Gamma(a_i', a_{i+1}))^{\frac{1}{2}}}.
\end{aligned}$$

By definition,

$$\Gamma(a_{i-1}, a_i) = \mu_{a_i}, \Gamma(a_i', a_{i+1}) = \mu_{a_{i+1}},$$

where  $\mu_i$  as in [F]. Thus,  $\mathfrak{ND}_n$  satisfies the recursive equation. Moreover, it is easy to see that

$$u_n = ND_{1,0,1,0,\dots,0,1,0,1},$$

where  $u_n$  is as in [F, equation 3.5, Page 551]. So  $\mathfrak{ND}_n$  satisfies the initial condition.  $\square$

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## References

- [BHMV] Blanchet, C.; Habegger, N.; Masbaum, G.; Vogel, P. *Topological quantum field theories derived from the Kauffman bracket*. Topology 34 (1995), no. 4, 883–927.
- [C] Comtet, L. *Advanced combinatorics: The art of finite and infinite expansions*. Revised and enlarged edition. D. Reidel Publishing Co., Dordrecht, 1974.
- [CP] Chen, Q.; Przytycki, J. *The Gram determinant of the type B Temperley-Lieb algebra*. Adv. in Appl. Math. 43 (2009), no. 2, 156–161.
- [FGG] Di Francesco, P.; Golinelli, O.; Guitter, E. *Meanders and the Temperley-Lieb algebra*. Comm. Math. Phys. 186 (1997), no. 1, 1–59.
- [F] Di Francesco, P. *Meander determinants*. Comm. Math. Phys. 191 (1998), no. 3, 543–583.

- [GS] Genauer, J.; Stoltzfus, N. W. *Explicit diagonalization of the Markov form on the Temperley-Lieb algebra*. Math. Proc. Cambridge Philos. Soc. 142 (2007), no. 3, 469–485.
- [KL] Kauffman, L. H.; Lins, S. L. *Temperley-Lieb recoupling theory and invariants of 3-manifolds*. Annals of Mathematics Studies, 134. Princeton University Press, Princeton, NJ, 1994.
- [K] Kirby, R. *A calculus for framed links in  $S^3$* . Invent. Math. 45 (1978), no. 1, 35–56.
- [L1] Lickorish, W. B. R. *Invariants for 3-manifolds from the combinatorics of the Jones polynomial*. Pacific J. Math. 149 (1991), no. 2, 337–347.
- [L2] Lickorish, W. B. R. *An introduction to knot theory*. Graduate Texts in Mathematics, 175. Springer-Verlag, New York, 1997.
- [L3] Lickorish, W. B. R. *Skeins and handlebodies*. Pacific J. Math. 159 (1993), no. 2, 337–349.
- [MV] Masbaum, G.; Vogel, P. *3-valent graphs and the Kauffman bracket*. Pacific J. Math. 164 (1994), no. 2, 361–381.
- [KS] Ko, K. H.; Smolinsky, L. *A combinatorial matrix in 3-manifold theory*. Pacific J. Math. 149 (1991), no. 2, 319–336.
- [RT] Reshetikhin, N.; Turaev, V. G. *Invariants of 3-manifolds via link polynomials and quantum groups*. Invent. Math. 103 (1991), no. 3, 547–597.
- [W] Witten, E. *Quantum field theory and the Jones polynomial*. Comm. Math. Phys. 121 (1989), no. 3, 351–399.